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THE FREE SURFACE ON A LIQUID FILLING A POROUS SLAB HEATED FROM ITS SIDES

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Abstract — The position of the free surface is calculated numerically for a porous slab which is partly filled with a liquid and differentially heated from its sides. A coordinate transformation is used to transform the original problem from the physical coordinate system to a non-orthogonal system where the free surface becomes a fixed straight line. The transformed problem is then solved using a finite-difference method. Results are obtained for Rayleigh numbers up to 1000.

NOMENCLATURE

- c specific heat
- D width of slab
- g acceleration due to gravity
- K permeability
- $\hat{k}$ unit vector in positive z-direction
- L depth of slab
- Nu Nusselt number
- P pressure
- Ra Rayleigh number
- T temperature
- ΔT $T(0,z)-T(D,z)$
- $\mathbf{u}$ nondimensional velocity
- $\mathbf{v}$ velocity, (v_x, v_z)
- x, z Cartesian coordinates
- z free surface, $h(x)$.

Greek symbols

- β volumetric thermal expansion coefficient
- γ L/D
- λ thermal conductivity coefficient
- ρ density of liquid
- ν kinematic viscosity
- θ nondimensional temperature
- ϕ velocity potential
- ψ stream function
- ∇ Nabla operator
- ∇^2 Laplace operator.

1. INTRODUCTION

THE STEADY fully saturated flow in a porous medium with heat conduction is considered for the case where the upper boundary consists of a free surface. The mathematical model consists of Poisson's equation for the stream function, the steady diffusion–convection equation for the temperature and a first-order equation for the free surface together with appropriate boundary conditions.

This problem has been analysed in ref. [1] by a small Rayleigh number perturbation for the case of a porous slab partly filled and differentially heated from the sides. A first-order expression for the free surface and a third-order approximation to the heat transfer across the slab were obtained.

There are two main difficulties in solving this problem numerically. Firstly the mathematical model is highly nonlinear, since both the diffusion–convection equation and the free surface equation are coupled through nonlinear terms; as a result the corresponding difference equations must be solved iteratively. The second difficulty is the irregularly shaped domain with the free surface which makes it difficult to apply finite-difference methods directly. We get around this difficulty by applying a coordinate transformation, which transforms the original domain into a rectangular domain where finite differences can be applied.

The resulting numerical model was solved for different values of the Rayleigh number, Ra, and we present numerical results for the free surface and the heat transfer and graphs of the streamlines and isotherms for Ra up to 1000.

The problem treated in this paper is a combination of free surface flow and heat convection. While many papers describing numerical procedures for either of these problems exist [2–6], this seems to be the first attempt in treating the combined problem. In none of the above papers was the case of heating from the sides treated; it seems that the first appearance of this problem was in ref. [1] where an analytic treatment was presented.

2. FORMULATION

We consider a long porous slab of width D which is partly filled with a liquid. At the reference temperature T_0 the liquid occupies the region $0 \leqslant x \leqslant D$, $-L \leqslant z < 0$, see Fig. 1; the line $z = 0$ is the free surface of the liquid. In general we represent this free surface by

$$z = h(x),$$

where $h(x)$ is an unknown function of x.

The flow field is given by the continuity equation

$$\nabla \cdot (\rho \mathbf{v}) = 0,$$

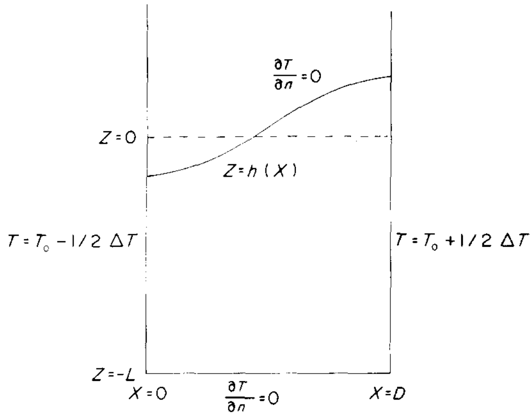

Fig. 1. Basic configuration.

and Darcy's law

$$\rho \mathbf{v} = -\frac{K}{\nu}(\nabla P + \rho g \hat{k}),$$

where ρ is the density, $\mathbf{v} = (v_x, v_z)$, the velocity vector, K the permeability, ν the kinematic viscosity, P the pressure, g the acceleration due to gravity, and $\hat{k}$ is the unit vector in the positive z-direction. The temperature field is given by the heat conduction equation

$$c\rho \mathbf{v} \cdot \nabla T = \lambda \nabla^2 T,$$

where c is the specific heat, T the temperature, and λ is the thermal conductivity coefficient. We also require an equation of state which gives the dependence of the density on temperature. For most liquids it is adequate to suppose that this relationship is linear; thus we will use

$$\rho = \rho_0[1 - \beta(T - T_0)],$$

where β is the volumetric thermal expansion coefficient and ρ_0 and T_0 are reference values.

At the free surface and at the bottom of the slab the heat flow is set to zero, while the two sides are kept at different temperatures. Thus the boundary conditions are

$$\frac{\partial T}{\partial z} = 0, \quad v_z = 0 \qquad \text{at } z = -L,$$

$$T = T_0 - \tfrac{1}{2}\Delta T, \quad v_x = 0 \text{ at } x = 0,$$

$$T = T_0 + \tfrac{1}{2}\Delta T, \quad v_x = 0 \text{ at } x = D,$$

$$\left.\begin{aligned} \frac{\partial T}{\partial z} - h'\frac{\partial T}{\partial x} &= 0 \\ v_z - h' v_x &= 0 \\ P &= 0 \end{aligned}\right\} \text{at } z = h(x).$$

The first boundary condition on the free surface $z = h(x)$ states that the normal temperature gradient there is zero, i.e. zero heat flux; the second condition is the kinematic boundary condition which expresses the fact that fluid particles which are at the free surface initially remain there. We suppose that the pressure at $z = h(x)$ is constant and since only ∇P appears in the differential equations, we can take this constant to be zero; this gives us the third condition. It should be noted that the concept of a sharp interface between a fully saturated and a completely unsaturated zone is a mathematical idealization; in reality there will be an intermediate zone which is partially saturated, and in this zone the mathematical model presented above will not be valid. However, in most practical cases the depth of this intermediate zone will be small compared to the other characteristic lengths, of the problem, such as the width of the slab. Thus results obtained from the present model should give very good approximate solutions for the complete problem.

We define nondimensional variables by

$$x^* = \frac{x}{D}, \quad z^* = \frac{z}{D}, \quad \nabla^* = D\nabla,$$

$$h^*(x^*) = \frac{h(x)}{D},$$

$$\mathbf{u} = \frac{\nu}{K\rho_0 g \beta \Delta T}\rho \mathbf{v},$$

$$\theta = \frac{T - T_0}{\Delta T},$$

and a nondimensional potential by

$$\phi = \left(\frac{P}{\rho_0 g} + z\right)\frac{1}{D\beta\Delta T}.$$

In the following nondimensional formulation we denote x^*, z^* and h^* by x, z and h for convenience. If we define a stream function ψ such that

$$u_x = \psi_z \quad \text{and} \quad u_z = -\psi_x,$$

the governing equations can be written in the form

$$\nabla^2 \psi = -\theta_x \tag{1}$$

$$\nabla^2 \theta = Ra(\psi_z \theta_x - \psi_x \theta_z) \tag{2}$$

where the Rayleigh number is given by

$$Ra = \frac{cK\rho_0 g D\varepsilon}{\lambda \nu},$$

and

$$\varepsilon = \beta \Delta T.$$

The corresponding boundary conditions become

$$\left.\begin{aligned} &\text{at } x = 0, && \theta = -\tfrac{1}{2}, \quad \psi = 0, \\ &x = 1, && \theta = \tfrac{1}{2}, \quad \psi = 0, \\ &z = -\frac{L}{D} = -\gamma, && \theta_z = 0, \quad \psi = 0, \\ &\text{at } z = h(x), && \begin{cases} -\psi_z + h'(\psi_x + \theta) = \dfrac{h'}{\varepsilon}, \\ \theta_z - h'\theta_x = 0, \\ \psi = 0. \end{cases} \end{aligned}\right\} \tag{3}$$

Since the mass is conserved we have also (in nondimensional form)

$$\int_0^1 h\,dx = \int_0^1 \int_{-\gamma}^{h(x)} \beta\Delta T\theta\,dz\,dx. \quad (4)$$

One of the questions we are interested in is how the presence of the free surface affects the heat transfer across the porous slab. At $x = 0$ it is given by

$$HT = \lambda\Delta T \int_{-\gamma}^{h} \theta_x(0,z)\,dz. \quad (5)$$

Here the integral is computed along the side wall $x = 0$ which forms an isotherm.

3. NUMERICAL PROCEDURE

Since we wish to apply finite-difference methods, it is convenient to use a coordinate transformation to transform the irregularly shaped domain $0 \leqslant x \leqslant 1$, $-\gamma \leqslant z \leqslant h(x)$ into a regularly shaped domain where all boundaries will coincide with grid lines. Hence we let

$$x_0 = x, \quad y = \frac{z-h(x)}{h(x)+\gamma}\gamma.$$

Thus we see that

$$z = -\gamma \to y = -\gamma \quad \text{and} \quad z = h(x) \to y = 0.$$

In the new coordinate system the boundary value problem, equations (1)–(5), becomes

$$D^2\psi = -\theta_x + h\frac{z+\gamma}{h+\gamma}\theta_y, \quad (6)$$

$$D^2\theta - Ra\frac{\gamma}{h+\gamma}[\psi_y\theta_x - \theta_y\psi_x] = 0, \quad (7)$$

where

$$D^2 = \frac{\partial^2}{\partial x^2} - 2h'\frac{z+\gamma}{h+\gamma}\frac{\partial^2}{\partial x \partial y} + \frac{h'^2(z+\gamma)^2+\gamma^2}{(h+\gamma)^2}\frac{\partial^2}{\partial y^2} + \frac{y+\gamma}{(h+\gamma)^2}[2h'^2 - h''(h+\gamma)]\frac{\partial}{\partial y},$$

and

$$\theta = -\tfrac{1}{2}, \quad \psi = 0 \quad \text{at } x = 0, \quad (8)$$

$$\theta = \tfrac{1}{2}, \quad \psi = 0 \quad \text{at } x = 1, \quad (9)$$

$$\theta_z = 0, \quad \psi = 0 \quad \text{at } y = -\gamma, \quad (10)$$

$$\psi = 0 \quad \text{at } y = 0, \quad (11)$$

$$\theta_z - \frac{h'(h+\gamma)}{\gamma(1+h'^2)}\theta_x = 0 \quad \text{at } y = 0, \quad (12)$$

$$h' = -\frac{\gamma\psi_y}{[(1/\varepsilon)-\theta](h+\gamma)-\gamma h'\psi_y} \quad \text{at } y = 0. \quad (13)$$

Here we have for convenience replaced x_0 by x. The mass conservation equation, equation (4), and the heat transfer, equation (5), become

$$\int_0^1 h\,dx = \frac{\varepsilon}{\gamma}\int_0^1 dx \int_{-\gamma}^{0} (h+\gamma)\theta\,dy, \quad (14)$$

$$HT = \frac{\lambda\Delta T}{\gamma}\int_{-\gamma}^{0} (h+\gamma)\theta_x(0,y)\,dy. \quad (15)$$

The region $0 \leqslant x < 1$, $-\gamma \leqslant y \leqslant 0$ is covered with a non-uniform finite-difference mesh defined by

$$0 = x_1 < x_2 < \cdots < x_{N+1} = 1,$$

and

$$-\gamma = y_1 < y_2 < \cdots < y_{M+1} = 0,$$

and we use the notation

$$\psi_{i,j} = \psi(x_i, y_j).$$

The non-uniform mesh was used since it is known that as the Rayleigh number Ra is increased, most of the variation in ψ and θ takes place near the boundaries. Hence a non-uniform mesh with the lines concentrated in these regions will give the best accuracy. Standard difference approximations are used, and the changes in mesh spacings are smooth enough that second-order accuracy is obtained throughout the region.

The numerical procedure for solving the boundary value problems, equations (6)–(14), consists of two parts: in the first part the field equations (6) and (7) together with the boundary conditions, equations (8)–(12), are solved for ψ and θ given the free surface function $h(x)$, while in the second part equations (13) and (14) are solved for $h(x)$ given ψ and θ. Since these two parts are coupled, it is necessary to use an iterative procedure. Thus if h_i^n is the nth approximation to $h(x)$, we solve equations (6) and (7) for $\psi_{i,j}^{n+1}$ and $\theta_{i,j}^{n+1}$. These values are then used in equations (13) and (14) which is solved for h_i^{n+1}. This iterative procedure is continued until

$$\max|h_i^{n+1} - h_i^n| < 10^{-5}.$$

With given h_i^n, $\psi_{i,j}^n$ and $\theta_{i,j}^n$ the procedure for solving equations (6) and (7) for $\psi_{i,j}^{n+1}$ and $\theta_{i,j}^{n+1}$ is as follows. We use second-order central difference formulae to replace equations (6) and (7) by corresponding difference equations; upwind–downwind formulae were not used. Since these algebraic equations are nonlinear, an iterative procedure must be used. The RHS of equation (6) is evaluated using h_i^n and $\theta_{i,j}^n$ and the resulting set of linear algebraic equations for an approximation to $\psi_{i,j}$ is swept once using a line SOR. These values are then used to evaluate ψ_y and ψ_x in equation (7) which is also swept once to produce an approximation to $\theta_{i,j}^{n+1}$. This procedure is repeated with the latest approximations to $\theta_{i,j}^{n+1}$ and $\psi_{i,j}^{n+1}$ being used to evaluate the RHS in equation (6) and ψ_y and ψ_x in equation (7) until the difference between two consecutive iterates is less than a given tolerance.

These values for $\psi_{i,j}^{n+1}$ and $\theta_{i,j}^{n+1}$ are now used in equations (13) and (19) which are then solved for h_i^{n+1}.

A second-order accurate difference approximation to equation (13) is

$$H_i^{n+1} = -[(\gamma/2\Delta)(\psi_{i,M-1}^{n+1} - 4\psi_{i,M}^{n+1})]/\{[(1/\varepsilon) - \theta_{i,M}^{n+1}] \times (h_i^{n+1}+\gamma) - (\gamma/2\Delta)H_i^n(\psi_{i,M-1}^{n+1} - 4\psi_{i,M}^{n+1})\}, \quad (16)$$

Table 1. $h(x)$ for different values of Ra with $\varepsilon = 0.5$ and $\gamma = 1.0$

Ra	0.0	0.2	0.4	0.6	0.8	1.0
$\ll 1$	−0.125	−0.093	−0.033	0.033	0.093	0.125
50	−0.0502	−0.02079	0.00828	0.03276	0.05181	0.06061
200	−0.035	−0.007	0.011	0.026	0.039	0.046
400	−0.025	−0.003	0.011	0.023	0.034	0.040
600	−0.021	−0.001	0.011	0.021	0.031	0.036
800	−0.017	−0.000	0.011	0.020	0.029	0.034
1000	−0.015	0.001	0.011	0.019	0.027	0.032

where $H_i = h_i'$ and we have assumed the two grid spacings next to $z = 0$ are equal so that

$$\Delta = z_{M+1} - z_M = z_M - z_{M-1}.$$

Thus

$$h_i^{n+1} = \int_0^{x_i} H_i^{n+1}\,dx + A, \tag{17}$$

where we obtain the integration constant A from the mass conservation equation (14)

$$A = \frac{\varepsilon}{\gamma}\int_0^1\left(\int_{-\gamma}^0 (h^{n+1}+\gamma)\theta^{n+1}\,dz\right)dx - \int_0^1\left(\int_0^x H^{n+1}\,d\sigma\right)dx. \tag{18}$$

Equations (16) and (17) form a nonlinear system for h_i^{n+1} when $\psi_{i,j}^{n+1}$ and $\theta_{i,j}^{n+1}$ are given. It was solved by direct iteration with the integrals in equations (17) and (18) evaluated using the trapezoidal rule. The convergence rate was increased by the use of overrelaxation.

When convergence is obtained, the heat transfer HT can be calculated by a numerical evaluation of equation (15).

Numerical solutions were also obtained for the case where the top surface is not free but a fixed lid is placed at $z = 0$. Then the appropriate boundary conditions are

$$\theta_z = 0, \quad \psi = 0 \quad \text{at } z = 0.$$

The numerical procedure is similar but considerably simpler since it is, of course, not necessary to carry out an outer iteration for the free surface.

4. RESULTS

In this section we shall present some of the results we obtained for different Rayleigh numbers. Experiments indicated that for $\gamma = 1$ a grid of 50 by 50 with non-uniform spacing in the x-direction. The calculations were repeated with different values of ε, but the dependence of ψ, θ and h on ε is rather weak so only results for $\varepsilon = 0.5$ are given. The calculations were carried out on a Cyber 170/835, and typical computing time was 200 s.

Table 2. Values of Nusselt number with $\varepsilon = 0.5$ and $\gamma = 1.0$

Ra	Nu	Nu (fitted top)
50	1.9	2.0>
200	5.4	5.3<
400	8.6	8.6>
600	11.3	11.3>
800	13.6	13.6>
1000	15.8	15.7<

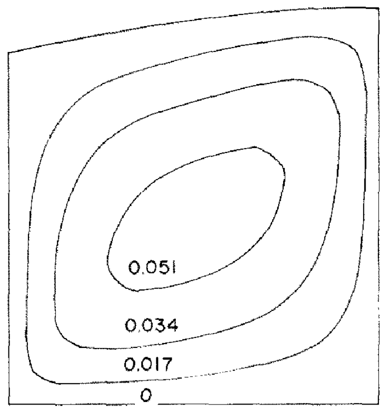

FIG. 2. Streamlines for $Ra = 50$ with $\varepsilon = 0.5$.

In Table 1 we present results for the free surface $z = h(x)$ for different values of Ra. We also include some results from ref. [1] which were obtained by a perturbation expansion in Ra; these values are symmetric about $x = 0.5$ since they are only accurate to first order in Ra.

Table 2 contains results for the Nusselt number defined by

$$Nu = \int_{-\gamma}^{h} \theta_x(0,z)\,dz = \frac{HT}{\lambda\Delta T}$$

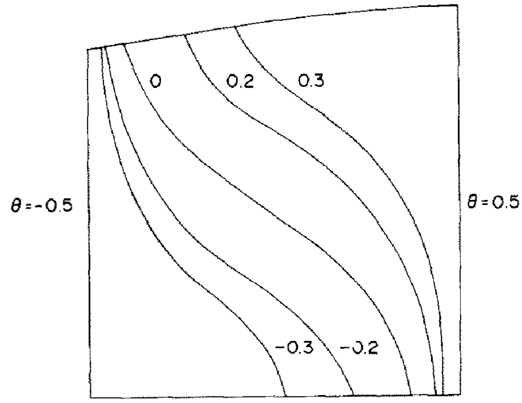

FIG. 3. Isotherms for $Ra = 50$ with $\varepsilon = 0.5$.

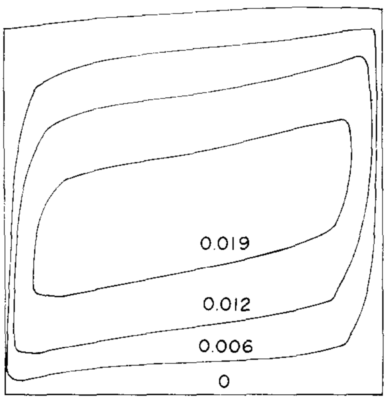

FIG. 4. Streamlines for $Ra = 1000$ with $\varepsilon = 0.5$.

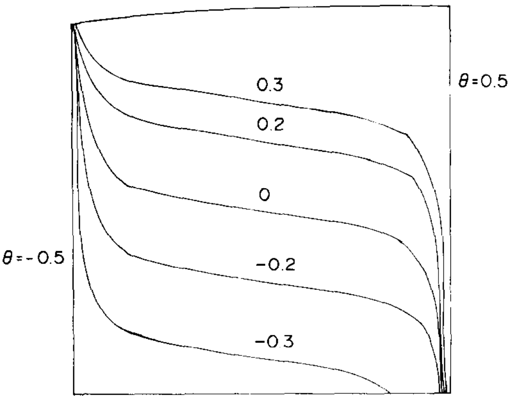

FIG. 5. Isotherms for $Ra = 1000$ with $\varepsilon = 0.5$.

for $\varepsilon = 0.5$; Nu was also calculated for the case of a fixed top. We have not been able to find values for Nu for the fixed top in the literature, so we do not have anything to compare our results with. In Figs. 2–5 streamlines and isotherms are plotted for Ra equal to 200 and 1000 with $\varepsilon = 0.5$ and $\gamma = 1$.

5. DISCUSSION

The study presented in this paper had two purposes. The first one was to develop a numerical procedure for solving such a highly nonlinear problem. The problem is nonlinear in two different ways; the field equations are coupled through the nonlinear convective terms in the heat conduction equations, and in addition the boundary conditions on the free surface are nonlinear. The results we obtained indicate that the numerical procedure works and is reasonably accurate. However, we have not been able to find any other published results to compare them with.

The second purpose of the study was to explore the effects of the free surface. The results in Table 1 show that the maximum depth of the saturated part of the slab increases by 8% for $Ra = 50$ while for $Ra = 1000$ the increase is 3% compared with the fixed top case. The conclusion we can draw from Table 2 is that the Nusselt number is not really dependent on the free surface; at least not at the accuracy with which these calculations were carried out.

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LA SURFACE LIBRE SUR UN LIQUIDE QUI REMPLIT UNE COUCHE POREUSE CHAUFFEE SUR SES COTES

Résumé—On calcule numériquement la position de la surface libre pour une couche poreuse qui est partiellement remplie par un liquide et qui est chauffée de façon différente sur ses cotés. Une transformation de coordonnées est utilisée pour transformer le système physique du problème original en un système non orthogonal où la surface libre devient une ligne droite fixe. Le problème transformé est alors résolu en utilisant une méthode aux différences finies. Des résultats sont obtenus pour des nombres de Rayleigh allant jusqu'à 1000.

DIE FREIE OBERFLÄCHE EINER FLÜSSIGKEIT, DIE IN EINEM VON DEN SEITEN HER BEHEIZTEN PORÖSEN STAB EINGESCHLOSSEN IST

Zusammenfassung—Die Lage der freien Oberfläche in einem porösen Stab, der teilweise mit Flüssigkeit gefüllt ist und dessen Seiten unterschiedlich beheizt werden, wird numerisch berechnet. Um das ursprüngliche Problem vom physikalischen in ein nicht-orthogonales Koordinatensystem zu übertragen, in dem die freie Oberfläche eine festgehaltene Gerade wird, führt man eine Koordinatentransformation durch. Das transformierte Problem wird mit Hilfe eines Differenzen-Verfahrens gelöst. Ergebnisse werden für Rayleigh-Zahlen bis 1000 erhalten.

СВОБОДНАЯ ПОВЕРХНОСТЬ ЖИДКОСТИ, ЗАПОЛНЯЮЩЕЙ ПОРИСТУЮ НАГРЕВАЕМУЮ С БОКОВЫХ СТОРОН ПЛИТУ

Аннотация—Для пористой плиты, частично заполненной жидкостью и неодинаково нагреваемой с боковых сторон, проведен численный расчет положения свободной поверхности. Методом преобразования координат исходная задача преобразуется из физической системы координат в неортогональную, где свободная поверхность представлена фиксированной прямой, и решается затем методом конечных разностей. Результаты получены для значений числа Рэлея до 1000.